FINITELY GENERATED SEMIGROUPS WITH REGULAR CONGRUENCE CLASSES(Algebras, Languages, Computations and their Applications)

SHOJI, KUNITAKA

数理解析研究所講究録
第 1562 巻 2007 年 127-131

FINITELY GENERATED SEMIGROUPS WITH REGULAR CONGRUENCE CLASSES *

KUNITAKA SHOJI(庄司 邦孝)
DEPARTMENT OF MATHEMATICS, SHIMANE UNIVERSITY
MATSUE, SHIMANE

In this paper, we give characterizations of finitely generated semigroups with regular congruence classes and finitely generated semigroups with finite congruence classes.

1 Presentations of semigroups

Definition. X is a finite alphabet, X^* is the set of all words over X. X^+ is the set of all non-empty words over X, that is, $X^+ = X^* - \{1\}$. Under juxtaposition, X^* is the free monoid with a set X of free generators and X^+ is the free semigroup with a set X of free generators.

A monoid M is *finitely generated* if there exists a finite set of X and there exists a surjective homomorphism of X^* to M which maps an empty word onto the identity element of M.

A semigroup S is *finitely generated* if there exists a finite set of X and there exists a surjective homomorphism of X^+ to S.

Definition (1) Let X be a finite alphabet and R a subset of $X^* \times X^*$. Then R is *string-rewriting system.*

(2) For $u, v \in X^*$, $(w_1, w_2) \in R$, $uw_1v \Rightarrow_R uw_2v$.

The congruence μ_R on X^* (or X^+) generated by $\Rightarrow_R$ is the Thue congruence defined by R.

(3) A monoid S has a *(finite)presentation* if there exists a (finite) set of X, there exists a surjective homomorphism ϕ of X^* to S and there exists a (finie) string-rewriting system R consisting of pairs of words over X such that the Thue congruence μ_R is the congruence $\{(w_1, w_2) \in X^* \times X^* \mid \phi(w_1) = \phi(w_2)\}$.

*This is an absrtact and the paper will appear elsewhere.

2 Semigroups with regular congruence classes

Definition. A semigroup S has *regular congruence classes* if there exists a finite set X and there exists a surjective homomorphism ϕ of X^+ to S such that for each words $w \in X^+$ $\phi^{-1}(\phi(w))$ is a regular language.

Definition. X is a finite set of alphabet, X^* is the set of words over X, L is a subset of X^*, is called a *language.* The *syntactic congruence* σ_L on X^* is defined by $w\sigma_L w'$ if and only if the sets $\{(x,y) \in X^* \times X^* \mid xwy \in L\}$, $\{(x,y) \in X^* \times X^* \mid xw'y \in L\}$ are equal to each other. The *syntactic monoid* of L is defined to be a monoid X^*/σ_L

Result 1. *Let L be a language over X. Then L is regular if and only if $Syn(L)$ is a finite monoid.*

Result 2. *Let L be a language of X^*. Then the following are equivalent :*

(1) *L is a σ_L-class in X^*.*

(2) *$xLy \cap L \neq ((x,y \in X^*) \Rightarrow xLy \subseteq L.$*

(3) *L is an inverse image $\phi^{-1}(m)$ of a homomorphism ϕ of X^* to a monoid M.*

Result 3. *For every finitely generated monoid M, there exist languages $\{L_m\}_{m \in M}$ of X^* such that M is embedded in the direct product of syntactic monoids.*

Definition. A monoid S is called *residually finite* if for each pair of elements $m, m' \in S$, there exists a conguence on S such that the factor monoid S/μ is finite and $(m, m') \notin \mu$.

Result 4. *If a finitely generated semigroup S has regular congruence classes, then M is residually finite.*

Definition. Let S be a finite generated semigroup. Let X be a finite set and there exists a surjective homomorphism ϕ of X^+ to S. Then the word problem of S is *decidable* if there exists an algorithsm to decide whether $\phi(w_1)$ is equal to $\phi(w_2)$ for each pair of words $w_1, w_2 \in X^+$.

Result 5. The word problem is decidable for finitely generated semigroups with regular congruence classes.

Exampe 1. *A finite semigroup S is semigroup with regular congruence classes.*

Definition. Let S be a semigroup. For any $s \in S$, let $\sigma_s = \{(a,b) \in S \times S \mid xay = s$ if and if $xby = s \ (x,y \in S^1)\}$.

Then σ_s is a congruence on S.

Theorem 1. *A finitely generated semigroup S has regular congruence classes if and only if for any $s \in S$, S/σ_s is a finite semigroup.*

Theorem 2. For a finitely generated semigroup S, it does not depend on presentations of S that S has regular congruence classes.

Theorem 3. *Let S be a finitely generated semigroup with regular congruence classes. Then a subgroup of S is finite.*

Example 2. Let X denote a finite alphabet and $w_1, \cdots, w_r \in X^+$ words over X. Let $I = X^*w_1X^* \cup \cdots \cup X^*w_rX^*$ be an ideal of the free semigroup X^+. Then The Rees factor semigroup X^+/I module I is a (unnecessarily finite) semigroup with regular congruence classes.

Result 6 . (1) *For every finite group G, there exists a regular language L of X^* such that G is isomorphic to $Syn(L)$.*

(2) *If a group G has regular congruence classes, then G is a finite.*

Theorem 4. Let S be a semigroup with regular congruence classes. If S is a completely (0-)simplesemigroup, then S is finite.

Exampe 3. *A residually finite semigroup S is not always a semigroup with regular congruence classes.*

3 Semigroups with finite congruence classes.

Definition. A semigroup S has *finite congruence classes* if there exists a finite set X and there exists a surjective homomorphism ϕ of X^+ to S such that for each words $w \in X^+$ $\phi^{-1}(\phi(w))$ is a finite set.

Theorem 5. *Let S be a semigroup with finite congruence classes. Then S has no idempotents except* 1 (*that is* , *S possibly has an idempotent*).

Theorem 6. *Let S be a semigroup with regular congruence classes. Then S is a semigroup with finite congruence classes if and only if for any $s \in S$, S/σ_s is a finite nilpotent semigroup with zero.*

Theorem 7. For a finitely generated semigroup S, it does not depend on presentations of S that S has finite congruence classes.

Theorem 8. Let S be a finitely presented semigroup with a string-rewriting system consisting of pairs of words of the same length. Then S is a semigroup with finite congruence classes.

Example 4. Let $X = \{x_1, x_2, \cdots, x_r\}$ and $\mathcal{R} = \{(x_i, x_j) \mid 1 \leq i < j \leq r\}$. Then $X^*/\mathcal{R}^*$ is a monoid with finite congruence classes and is the commutative free monoid.

Theorem 9. Let S be a finitely generated semigroup with a non-overlapping string-rewriting system. Then S is a semigroup with finite congruence classes.

Theorem 10. Any finitely generated subsemigroup of the free semigroup is a semigroup with finite congruence classes.

Example 5 There exists a finitely generated subsemigroup S of the free semigroup which is a semigroup with finite congruence classes but does not have a finite presentation.

Actually, let $X = \{A, B, V, W\}$. Then $V(AB)^n W = VA(BA)^{n-1}W$ in X^+. So the finitely generated subsemigroup $< V, VA, AB, BA, W, BW >$ is isomophic to non-finitely presented semigroup $Y =< a, b, c, d, e, f \mid ac^n e = bd^{n-1} f (n = 0, 1, 2, \cdots) >$. By theorem 10, the semigroup is a semigroup with finite congruence classes.

The

Theorem 11. *Any semigroup with either $C(3)$ or $C(2) + T(4)$ has finite congruence classese.* (Refer to [1],[3],[4] and [7] for the conditions $C(p)$, $T(q)$.)

References

[1] P.A. Cummings and R.Z. Goldstein, *Solvable word problems in semigroups*, Semigroup Forum **50**(1995), 243-246.

[2] E. A. Golubov, *Finite separability in semigroups*, (Russian) Sibirsk. Mat. Ž. **11**(1970), 1247-1263.

[3] P.M. Higgins, *Techniques of semigroup theory*, Oxford University Press, New York, 1992.

[4] P. Hill, S.J. Pride and A.D. Vella, *On the $T(q)$-conditions of small cancellation theory*, Isral J. Math. **52**(1985), 293-304.

[5] T. Mitoma and Shoji, *Syntactic monoids and languages*, urikaisekikenkyusho kokyuroku **1437**(2005), 11-16.

[6] J.E. Pin, *Varieties of formal languages*, North Oxford Academic Publishers, London, 1986.

[7] J.H. Remmers, *On the geometry of semigroup presentations*, Adv. in Math. **36**(1980), 283-296.